map $\phi \times \psi : W \times U \longrightarrow X \times Y$ by the formula

$$\phi \times \psi(w, u) = (\phi(w), \psi(u)).$$

Of course, $W \times U$ is an open set in $\mathbf{R}^k \times R^l = \mathbf{R}^{k+l}$, and it is easy to check that $\phi \times \psi$ is a local parametrization of $X \times Y$ around (x, y). (Check this point carefully, especially verifying that $(\phi \times \psi)^{-1}$ is a smooth map on the not-necessarily-open set $X \times Y \subset \mathbf{R}^{M+N}$). Since this map is a local parametrization around an arbitrary point $(x, y) \in X \times Y$, we have proved:

Theorem. If X and Y are manifolds, so is $X \times Y$, and $\dim X \times Y = \dim X + \dim Y$.

We mention another useful term here. If X and Z are both manifolds in $\mathbf{R}^N$ and $Z \subset X$, then Z is a *submanifold* of X. In particular, X is itself a submanifold of $\mathbf{R}^N$. Any open set of X is a submanifold of X.

The reader should be warned that the slothful authors will often omit the adjective "smooth" when speaking of mappings; nevertheless, smoothness is virtually always intended.

EXERCISES

1. If $k < l$ we can consider $\mathbf{R}^k$ to be the subset $\{(a_1, \ldots, a_k, 0, \ldots, 0)\}$ in $\mathbf{R}^l$. Show that smooth functions on $\mathbf{R}^k$, considered as a subset of $\mathbf{R}^l$, are the same as usual.

*2. Suppose that X is a subset of $\mathbf{R}^N$ and Z is a subset of X. Show that the restriction to Z of any smooth map on X is a smooth map on Z.

*3. Let $X \subset \mathbf{R}^N$, $Y \subset \mathbf{R}^M$, $Z \subset \mathbf{R}^L$ be arbitrary subsets, and let $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ be smooth maps. Then the composite $g \circ f : X \longrightarrow Z$ is smooth. If f and g are diffeomorphisms, so is $g \circ f$.

4. (a) Let B_a be the open ball $\{x : |x|^2 < a\}$ in $\mathbf{R}^k$. ($|x|^2 = \sum x_i^2$) Show that the map

$$x \longrightarrow \frac{ax}{\sqrt{a^2 - |x|^2}}$$

is a diffeomorphism of B_a onto $\mathbf{R}^k$. [HINT: Compute its inverse directly.]

(b) Suppose that X is a k-dimensional manifold. Show that every point in X has a neighborhood diffeomorphic to all of $\mathbf{R}^k$. Thus local parametrizations may always be chosen with all of $\mathbf{R}^k$ for their domains.

*5. Show that every k-dimensional vector subspace V of $\mathbf{R}^N$ is a manifold diffeomorphic to $\mathbf{R}^k$, and that all linear maps on V are smooth. If $\phi: \mathbf{R}^k \to V$ is a linear isomorphism, then the corresponding coordinate functions are linear functionals on V, called *linear coordinates*.

6. A smooth bijective map of manifolds need not be a diffeomorphism. In fact, show that $f: \mathbf{R}^1 \to \mathbf{R}^1, f(x) = x^3$, is an example.

7. Prove that the union of the two coordinate axes in $\mathbf{R}^2$ is not a manifold. (HINT: What happens to a neighborhood of 0 when 0 is removed?)

8. Prove that the paraboloid in $\mathbf{R}^3$, defined by $x^2 + y^2 - z^2 = a$, is a manifold if $a > 0$. Why doesn't $x^2 + y^2 - z^2 = 0$ define a manifold?

9. Explicitly exhibit enough parametrizations to cover $S^1 \times S^1 \subset \mathbf{R}^4$.

10. "The" *torus* is the set of points in $\mathbf{R}^3$ at distance b from the circle of radius a in the xy plane, where $0 < b < a$. Prove that these tori are all diffeomorphic to $S^1 \times S^1$. Also draw the cases $b = a$ and $b > a$; why are these not manifolds?

11. Show that one cannot parametrize the k sphere S^k by a single parametrization. [HINT: S^k is compact.]

*12. Stereographic projection is a map π from the punctured sphere $S^2 - \{N\}$ onto $\mathbf{R}^2$, where N is the north pole (0, 0, 1). For any $p \in S^2 - \{N\}$, $\pi(p)$ is defined to be the point at which the line through N and p intersects the xy plane (Figure 1-6). Prove that $\pi: S^2 - \{N\} \to \mathbf{R}^2$ is a diffeomorphism. (To do so, write π explicitly in coordinates and solve for π^{-1}.)

Note that if p is near N, then $|\pi(p)|$ is large. Thus π allows us to think of S^2 a copy of $\mathbf{R}^2$ compactified by the addition of one point "at infinity." Since we can define stereographic projection by using the

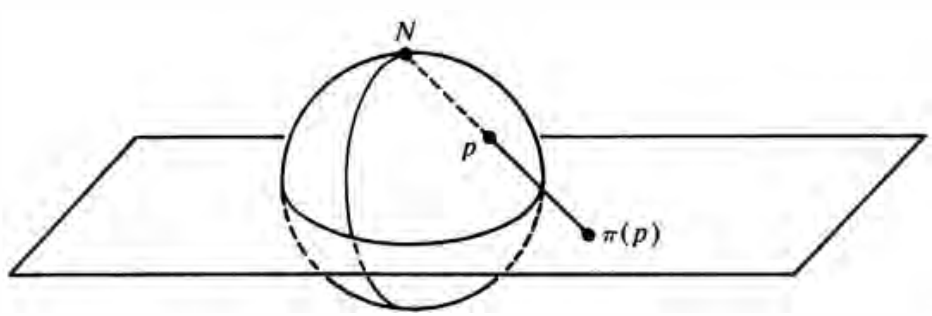

Figure 1-6

south pole instead of the north, S^2 may be covered by two local parametrizations.

***13.** By generalizing stereographic projection define a diffeomorphism $S^k - \{N\} \longrightarrow \mathbf{R}^k$.

***14.** If $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$ are smooth maps, define a *product map* $f \times g : X \times Y \longrightarrow X' \times Y'$ by

$$(f \times g)(x, y) = (f(x), g(y)).$$

Show that $f \times g$ is smooth.

15. Show that the projection map $X \times Y \longrightarrow X$, carrying (x, y) to x, is smooth.

***16.** The *diagonal* Δ in $X \times X$ is the set of points of the form (x, x). Show that Δ is diffeomorphic to X, so Δ is a manifold if X is.

***17.** The *graph* of a map $f: X \longrightarrow Y$ is the subset of $X \times Y$ defined by

$$\text{graph}\,(f) = \{(x, f(x)) : x \in X\}.$$

Define $F: X \longrightarrow \text{graph}\,(f)$ by $F(x) = (x, f(x))$. Show that if f is smooth, F is a diffeomorphism; thus graph (f) is a manifold if X is. (Note that $\Delta = \text{graph (identity)}$.)

***18.** (a) An extremely useful function $f: \mathbf{R}^1 \longrightarrow \mathbf{R}^1$ is

$$f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Prove that f is smooth.

(b) Show that $g(x) = f(x - a)g(b - x)$ is a smooth function, positive on (a, b) and zero elsewhere. (Here $a < b$.) Then

$$h(x) = \frac{\int_{-\infty}^{x} g\,dx}{\int_{-\infty}^{\infty} g\,dx}$$

is a smooth function satisfying $h(x) = 0$ for $x < a$, $h(x) = 1$ for $x > b$, and $0 < h(x) < 1$ for $x \in (a, b)$.

(c) Now construct a smooth function on $\mathbf{R}^k$ that equals 1 on the ball of radius a, zero outside the ball of radius b, and is strictly between 0 and 1 at intermediate points. (Here $0 < a < b$.)